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On Generalized Lee Weights for Codes over \mathbb{Z}_4 *

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1 Introduction

For a linear code over a finite field, Helleseth, Klove and Mykkeltveit [9] introduced the generalized Hamming weights while studying the weight distribution of irreducible cyclic codes and later Wei ([18]) rediscovered the idea of generalized Hamming weights. After that a lot of papers dealing with the weights have been published (cf. [17] etc.). Recently, the generalized Hamming weights for codes over \mathbb{Z}_4 have been defined and studied, see [1], [19], [20], [3] and [10] for example.

In this note, we shall define a type of generalized Lee weights for codes over \mathbb{Z}_4 and give some fundamental results.

A *linear code* of length n over \mathbb{Z}_4 is a \mathbb{Z}_4 -submodule of \mathbb{Z}_4^n . For a linear code C of length n over \mathbb{Z}_4 , we define the *rank* of C , denoted by $\text{rank}(C)$, by the minimum number of generators of C . It is known that a linear code C of length n over \mathbb{Z}_4 is permutation-equivalent to a linear code with generator matrix of the form

$$(1) \quad \begin{pmatrix} I_{k_1} & X & Y \\ 0 & 2I_{k_2} & 2Z \end{pmatrix},$$

where X and Z are binary matrices and Y is a \mathbb{Z}_4 -matrix. In this case, it finds that $|C| = 4^{k_1} 2^{k_2}$ and $\text{rank}(C) = k_1 + k_2$. We shall define a code with a generator matrix of the form in 1 as being of type $\{k_1, k_2\}$.

For a vector $\mathbf{x} \in \mathbb{Z}_4^n$, we denote the *Hamming weight* and *Lee weight* by $\text{wt}(\mathbf{x})$ and $\text{L-wt}(\mathbf{x})$, respectively.

For a linear code C of length n over \mathbb{Z}_4 , let $A(C)$ be the $|C| \times n$ array of all codewords in C . It is well-known that each column of $A(C)$ corresponds to the following three cases: (i)

*This work is jointed with Steven T. Dougherty and Manish Gupta.

the column contains only 0 (ii) the column contains 0 and 2 equally often (iii) the column contains all elements of \mathbb{Z}_4 equally often (cf. [20]). For the three columns (i), (ii) and (iii), we define the *Lee weights* of these columns by 0, 2 and 1 respectively. Thus we define the *Lee weight* $\text{wt}_L(C)$ of C by the sum of the Lee weights of all columns of $A(C)$. For example, if

$$C = \{(0, 0, 0), (1, 0, 1), (2, 0, 2), (3, 0, 3), (0, 2, 2), (1, 2, 3), (2, 2, 0), (3, 2, 1)\},$$

then $\text{wt}_L(C) = 1 + 2 + 1 = 4$. We remark that if C is generated by only one vector \mathbf{x} , then the Lee weight $\text{wt}_L(C)$ corresponds to the original Lee weight $\text{L-wt}(\mathbf{x})$ of \mathbf{x} . Then we have the following theorem.

Theorem 1.1 *Let C be a linear code C of length n over \mathbb{Z}_4 with type $4^{k_1}2^{k_2}$. Then we have*

$$\begin{aligned} \text{wt}_L(C) &= \frac{1}{4^{k_1-1}2^{k_2}} \sum_{\mathbf{x} \in C} (\text{L-wt}(\mathbf{x}) - \text{wt}(\mathbf{x})) \\ &= \frac{1}{4^{k_1-1}2^{k_2}} \sum_{\mathbf{x} \in C} |\{i : x_i = 2\}|. \end{aligned}$$

Now, for $1 \leq r \leq \text{rank}(C)$, we define the r -th *generalized Lee weight with respect to rank* (GLWR) $d_r^L(C)$ of C as follows:

$$d_r^L(C) := \min\{\text{wt}_L(D) : D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with } \text{rank}(D) = r\}.$$

We note that $d_1^L(C)$ corresponds to the minimum Lee weight of C .

2 Bounds for GLWR

In this section, we give some bounds for GLWR of linear codes over \mathbb{Z}_4 .

Lemma 2.1 *If C is a linear code of length n over \mathbb{Z}_4 with $\text{rank}(C) = 2$, then there exists a codeword $0 \neq \mathbf{v} \in C$ such that $\text{L-wt}(\mathbf{v}) \leq \text{wt}_L(C)$.*

Using the above lemma, we have the following result.

Theorem 2.2 *Let C be a linear code of length n over \mathbb{Z}_4 with $\text{rank}(C) \geq 2$. Then we have $1 \leq d_1^L(C) \leq d_2^L(C)$.*

In [11], the r th generalized Hamming weight with respect to rank (GHWR) of a linear code C is defined by

$$d_r^H(C) := \min\{|\text{Supp}(D)| : D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with } \text{rank}(D) = r\},$$

where $\text{Supp}(D) := \cup_{\mathbf{x} \in D} \text{supp}(\mathbf{x})$. We remark that

$$(2) \quad d_r^L(C) \leq 2d_r^H(C).$$

The following lemma is called the *generalized Singleton bound* for linear codes over \mathbb{Z}_4 (see

Lemma 2.3 *Let C be a linear code of length n over \mathbb{Z}_4 . Then, for any r , $1 \leq r \leq \text{rank}(C)$,*

$$d_r^H(C) \leq n - \text{rank}(C) + r.$$

Now, we give a similar type bound for GLWR.

Theorem 2.4 *For a linear code C of length n over \mathbb{Z}_4 and any r , $1 \leq r \leq \text{rank}(C)$,*

$$\left\lfloor \frac{d_r^L(C) - 2r + 1}{2} \right\rfloor \leq n - \text{rank}(C).$$

Remark 2.5 In [7] and [15], it is shown that for a linear code C of length n over \mathbb{Z}_4 with minimum Lee weight d_L ,

$$\left\lfloor \frac{d_L - 1}{2} \right\rfloor \leq n - \text{rank}(C).$$

Since $d_L = d_1^L(C)$, the bound in Theorem 2.4 is a generalization of the above bound.

If a linear code C of length n over \mathbb{Z}_4 meets the bound in Theorem 2.4 for r , that is, $\left\lfloor (d_r^L(C) - 2r + 1)/2 \right\rfloor = n - \text{rank}(C)$, then we shall call the code C as *r -th maximum Lee distance separable with respect to rank* (r -th MLDR) code. Similarly if a linear code C of length n over \mathbb{Z}_4 meets the bound in Lemma 2.3 for r , that is, $d_r^H(C) = n - \text{rank}(C) + r$, then the code C is called *r -th maximum Hamming distance separable with respect to rank* (r -th MHDR) code. Now we shall give a connection between r -th MLDR codes and r -th MHDR codes.

Lemma 2.6 *If C is an r -th MLDR code, then $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$.*

Theorem 2.7 *Let C be a linear code C of length n over \mathbb{Z}_4 . If C is an r -th MLDR code, then C is an r -th MHDR code.*

Theorem 2.8 *Let C be an r -th MHDR code of length n over \mathbb{Z}_4 . C is an r -th MLDR code if and only if $d_r^L(C) = 2d_r^H(C) - 1$ or $2d_r^H(C)$.*

It is known that if C is a linear code of length n over \mathbb{Z}_4 with minimum Hamming weight d_H and minimum Lee weight d_L , then

$$(3) \quad d_H \geq \left\lceil \frac{d_L}{2} \right\rceil$$

(cf. [14]). In [16], they have proved the following Griesmer type bound for linear codes over finite quasi-Frobenius rings.

Lemma 2.9 *Let C be a linear code of length n over \mathbb{Z}_4 with $\text{rank}(C) = k$ and minimum Hamming weight d_H . Then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H}{2^i} \right\rceil.$$

Using (3) and Lemma 2.9, we have the following Griesmer type bound for minimum Lee weights of linear codes over \mathbb{Z}_4 .

Proposition 2.10 *Let C be a linear code of length n over \mathbb{Z}_4 with $\text{rank}(C) = k$ and minimum Lee weight d_L . Then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{\lceil d_L/2 \rceil}{2^i} \right\rceil.$$

Now we have a generalized Griesmer type bound for GLWR.

Theorem 2.11 *For a linear code C of length n over \mathbb{Z}_4 and any r , $1 \leq r \leq \text{rank}(C)$, we have*

$$d_r^L(C) \geq \sum_{i=0}^{r-1} \left\lceil \frac{\lceil d_1^L(C)/2 \rceil}{2^i} \right\rceil.$$

Let C be a linear code C of length n over \mathbb{Z}_4 . From the definitions of GLWR and GHWR, we have

$$(4) \quad d_r^H \geq \left\lceil \frac{d_r^L}{2} \right\rceil$$

for any r . We define the *socle* of C as follows:

$$\text{Soc}(C) := \{\mathbf{x} \in C \mid 2\mathbf{x} = \mathbf{0}\}.$$

It is known that if C is a linear code C of length n over \mathbb{Z}_4 with $\text{rank}(C) = k$ and minimum hamming weight d_H , then $\text{Soc}(C)$ is isomorphic to a binary $[n, k, d]$ code (cf. [11]).

Lemma 2.12 ([11]) *For any r , $1 \leq r \leq \text{rank}(C)$, we have*

$$d_r^H(C) = d_r^H(\text{Soc}(C)).$$

Using the above lemma and Theorem 3.19 (p. 35 in [5]), the lemma follows:

Lemma 2.13 *Let C be a linear code C of length n over \mathbb{Z}_4 with $\text{rank}(C) = k$. Then*

$$n \geq d_r^H(C) + \sum_{i=1}^{k-r} \left\lceil \frac{d_r^H(C)}{2^i(2^i - 1)} \right\rceil,$$

for any r , $1 \leq r \leq k$.

Now we have a generalized Griesmer type bound for GLWR.

Theorem 2.14 *Let C be a linear code C of length n over \mathbb{Z}_4 with $\text{rank}(C) = k$. Then*

$$n \geq \left\lceil \frac{d_r^L(C)}{2} \right\rceil + \sum_{i=1}^{k-r} \left\lceil \frac{\left\lceil \frac{d_r^L(C)}{2} \right\rceil}{2^i(2^i - 1)} \right\rceil,$$

for any r , $1 \leq r \leq k$.

References

- [1] A. Ashikhmin, On generalized Hamming weights for Galois ring linear codes, *Designs, Codes and Cryptography*, **14** (1998) pp. 107–126.
- [2] *Handbook of coding theory Vol. I* (Edited by V. Pless, W. Huffman and R. Brualdi), North-Holland, Amsterdam, 1998.
- [3] S. T. Dougherty and K. Shiromoto, Maximum distance codes over rings of order 4, *IEEE Trans. Inform. Theory*, **47** (2001) pp. 400–404.
- [4] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane and P. Solé, The \mathbb{Z}_4 -linearity of Kerdock, Preparata, Goethals and related codes, *IEEE Trans. Inform. Theory*, **40** (1994) pp. 301–319.
- [5] T. Helleseth, T. Klove and J. Mykkeltveit, The weight distribution of irreducible cyclic codes with block lengths $n_1 \left(\frac{q^l-1}{N} \right)$, *Discrete Mathematics*, **18** (1979) pp. 179–211.
- [6] T. Helleseth and K. Yang, Further results on generalized Hamming weights for Goethals and Preparata codes over \mathbb{Z}_4 , *IEEE Trans. Inform. Theory*, **45** (1999) pp. 1255–1258.
- [7] H. Horimoto and K. Shiromoto, On generalized Hamming weights for codes over finite chain rings, *Lecture Notes in Computer Science*, **2227** (2001) pp. 141–150.
- [8] B. Hove, Generalized Lee weights for codes over \mathbb{Z}_4 , *Proc. IEEE Int. Symp. Inf. Theory*, p. 203, Ulm, Germany, (1997).
- [9] F. J. MacWilliams and N. J. A. Sloane, *The theory of error-correcting codes*, North-Holland, Amsterdam 1977.
- [10] E. M. Rains, Optimal self-dual codes over \mathbb{Z}_4 , *Discrete Mathematics*, **203** (1999), pp. 215–228.
- [11] K. Shiromoto, A basic exact sequence for the Lee and Euclidean weights of linear codes over \mathbb{Z}_ℓ , *Linear Algebra and its Applications*, **295**, pp. 191–200, 1999.

- [12] K. Shiromoto and L. Storme, A Griesmer bound for codes over finite quasi-Frobenius rings, *Electronic Notes in Discrete Mathematics* (to appear).
- [13] M. A. Tsfasman and S. G. Vladut, Geometric approach to higher weights, *IEEE Trans. Inform. Theory*, **41** (1995) pp. 1564–1588.
- [14] V. K. Wei, Generalized Hamming weights for linear codes, *IEEE Trans. Inform. Theory*, **37** (1991) pp. 1412–1418.
- [15] K. Yang, T. Helleseht, P. V. Kumar and A. G. Shanbhang, On the weights hierarchy of Kardock codes over \mathbb{Z}_4 . *IEEE Trans. Inform. Theory*, **42** (1996) pp. 1587–1593.
- [16] K. Yang and T. Helleseht, On the weight hierarchy of Preparata codes over \mathbb{Z}_4 , *IEEE Trans. Inform. Theory*, **43** (1997) pp. 1832–1842.